Approximate Minimum Energy Control

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A computationally simple approximation to the minimum energy controller for a linear system is achieved by reducing the complexity of the mathematical model of the system. After specifying the criterion for identifying an optimal model of lower order, the required model boundary conditions are deduced. Controller performance is expressed as a scalar valued functional of the model coefficient matrices, and a computational algorithm is suggested for evaluating the best model. The suboptimal control is then expressed in terms of the derived model parameters.

1. Introduction

In a number of diverse applications a designer seeks a controller that will cause the state of a dynamic system to be transferred from a given initial value to a specified final value in a fixed time interval with minimum expenditure of energy. For example, one might seek the minimum energy thrust program of a booster from launch until the point of decoupling. Trajectory analysis of a maneuverable space vehicle may also introduce problems of this general type.

If the system to be controlled is modelled with a set of linear ordinary differential equations and if the amplitudes of the input signals are unconstrained, the minimum energy control signal vector may be displayed explicitly.[1,2] If in addition the model has constant coefficients, the control may be expressed as a linear algebraic operation on the transition matrix of a related linear system.[3] Because of the nature of this auxiliary system, such control is sometimes referred to as adjoint steering.

In principle, the problem is thus solved. In applications, however, additional difficulties may arise. If the dynamic system is to be self contained, it must possess the capability of generating its own control. For the problem posed here, this necessitates either storage of the complete control time function or an ability to generate the control on-line. The former alternative requires a great amount of memory capacity, whereas the latter alternative would require a great amount of on-line computation if the system model were complicated.

Consequently, adjoint steering is most easily mechanized when the model of the dynamic system is of low order. Yet, in many applications just the reverse is true. A flexible booster is best described as a distributed parameter system. Thus, the relevant dynamics should be expressed by a set of partial differential equations. To avoid the complexities inherent in distributed systems, the designer may derive a set of differential equations for the dynamic modes of the vehicle. This has the natural advantage that the resulting mathematical model is phrased in terms of a set of ordinary differential equations. It also has the distinct drawback that any finite dimensional model must be only an approximation to the original distributed system.

In order to satisfy the exigencies introduced by the limited on-line computational capacity and the high dimension of the model state vector, the designer is often forced simply to neglect all dynamic modes beyond some arbitrary frequency. The optimal control is then obtained from this reduced model. The flaw in such a procedure is not only that the resulting control is suboptimal; but, in addition, one or more of the neglected modes may be excited by the control, and vehicle instability can result.

Other approximations are also made for the sake of mechanizational simplicity; for example, time varying model parameters may be replaced by constants. Questions related to performance deterioration are often ignored in such approximations, and classical function approximation techniques are used instead.

In this paper the problem of selecting a low-order linear constant coefficient model for a high-order and possibly time variable linear system is considered. This problem is posed within the context of finding a suitable approximation to the minimum energy control for the original system. The result of the analysis is a computational procedure for evaluating a simple model which may be used to reduce the requisite on-line controller complexity. The simple model may also be used in simulation studies to evaluate qualitatively the system sensitivity to disturbances and variations in initial conditions.

The method of analysis is intimately related to that employed by Heffes and Sarachik.[4] In this reference a time variable unforced system was approximated by a constant coefficient model of lower dimension. Heavy reliance was placed on certain orthogonality properties of the approximation. Here, a control input must be used to satisfy the design objectives, but as will become clear, these same orthogonality relations may be employed in evaluating the approximate model.

2. Problem Description

Suppose that the system to be controlled can be described by a set of linear differential equations

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),\ t_0 \leq t \leq T, \qquad x(t_0) = x_0 \quad (1)$$

where $A(t)$ and $B(t)$ are $n \times n$ and $n \times r$ matrices, respectively, with bounded and piecewise continuous elements. The n vector $x(t)$ represents the system state, and the actuating signal $u(t)$ is an r vector.

The objective of the controller is the transfer of the system state from x_0 to a specified terminal state x_d at time T with minimum expenditure of energy. The energy necessary to make this transfer will be assumed proportional to the

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square of the norm of the actuating signal where

$$\|u\|^2 = \int_{t_0}^{T} u(s)'u(s)ds \tag{2}$$

It will further be assumed that the system described by Eq. (1) is controllable and that the magnitude of $u(t)$ is unconstrained. Let $\Phi(t_1,t_2)$ be the transition matrix of this system and define‡

$$W(t_2,t_1) = \int_{t_1}^{t_2} \Phi(t_2,s)B(s)B'(s)\Phi'(T,s)ds \tag{3}$$

It is well known that the control

$$\hat{u}(t) = B'(t)\Phi'(T,t)W(T,t_0)^{-1}[x_d - \Phi(T,t_0)x_0] \tag{4}$$

is the unique control of minimum norm which transfers $x(t)$ to x_d at time T^1.

Equation (4) has an interesting interpretation. Since $\Phi'(T,t)$ is the transition matrix of the system adjoint to that of Eq. (1), $\Phi'(T,t)W(T,t_0)^{-1}[x_d - \Phi(T,t_0)x_0]$ is the response of the adjoint system to an initial condition p_0 where

$$p_0 = W(T, t_0)^{-1}[x_d - \Phi(T,t_0)x_0] \tag{5}$$

Consequently, $\hat{u}(t)$ is sometimes called adjoint steering. The optimal trajectory, $\hat{x}(t)$ may be evaluated from Eqs. (1) and (4):

$$\hat{x}(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)B(s)\hat{u}(s)ds = \Phi(t,t_0)x_0 + W(t,t_0)p_0 \tag{6}$$

In principle Eq. (4) gives the solution to the problem posed. The n vector p_0 may be computed off-line and $\Phi'(T,t)$ may be generated recursively on-line from its defining equation. It is at this point that dimensionality problems may present themselves; for example, Eq. (1) may be a linearized model of a flexible booster. In order to describe adequately the dynamic behavior of the booster, the dimension of the state variable may be quite high. To determine $\Phi'(T,t)$ accurately the up-dating time increment must be small, yet to compute $\Phi(T,t)$ on-line the computation time must be small. These mutually contradictory demands can be met only if the dimension of the state vector is low. Often this forces the designer to reduce the dimension of the model by arbitrarily neglecting components of the state vector beyond a specified point. Because of this arbitrary restriction on the dimension of the state vector, the terminal conditions of the problem will not normally be satisfied. This may not cause an unacceptable deterioration in performance if the system trajectory derived from analysis of the simple model is close to the optimal one. Unfortunately, this is not always the case. While the portion of model thus neglected may represent the high-frequency dynamics of the system, it is precisely these modes which may be resonant to the input derived from the lower order model.

Another step in the simplification of the control often is accomplished by replacing the coefficient matrices in Eq. (1) by constant approximations. This simplifies Eq. (4), for the adjoint system now has constant coefficients and $\Phi(T,t)$ may be expressed as a matrix exponential.

As the foregoing discussion indicates, applications utilizing the minimum energy control often require that the designer use a much simplified model of the system to be controlled. He needs assurance, however, that the results he derives from the analysis of response of this reduced model will correlate well with the actual system response.

To be specific, suppose that the reduced model of Eq. (1) has the form

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t),\ t_0 \le t \le T,\ \tilde{x}(t_0) = \tilde{x}_0 \tag{7}$$

where $\tilde{x}(t)$ is the m-dimensional state and $u(t)$ is the r-dimensional actuating signal. The constant coefficient matrices $\tilde{A}$ and $\tilde{B}$ are of compatible order. It will be assumed that the system described by Eq. (7) is controllable and that $m < n$.

‡ For any matrix A, A' denotes A transpose.

The lower order model described by Eq. (7) is an intermediary through which the designer seeks to obtain an approximately optimal system. The minimum norm control for the reduced order model will quite clearly be simpler to mechanize than is Eq. (4). To be useful in this application, however, this new control must also be satisfactory when employed in the original system.

Suppose that $\tilde{x}(T)$ is constrained to be $\tilde{x}_d$. The minimum norm control of Eq. (7) is given by

$$\tilde{u}(t) = \tilde{B}'\tilde{\Phi}'(T,t)\tilde{W}(T,t_0)^{-1}[\tilde{x}_d - \tilde{\Phi}(T,t_0)\tilde{x}_0] \tag{8}$$

where

$$\tilde{\Phi}(T,t) = \exp\tilde{A}(T - t)$$
$$\tilde{W}(t_2,t_1) = \int_{t_1}^{t_2} \tilde{\Phi}(t_2,s)\tilde{B}\tilde{B}'\tilde{\Phi}'(T,s)ds \tag{9}$$

Usually the dimension of the reduced system is selected after a preliminary study defines the permissible on-line computational complexity of the controller. Two questions still remain. For a given reduced model, how should the initial and terminal conditions be chosen so that the control of Eq. (8) yields good performance in the original system? Then, for a given set of boundary conditions on Eq. (7), how should the dynamics of the model be selected?

Denote by $x^*(t)$ the solution to Eq. (1) with the control given by Eq. (8). Then

$$x^*(t) = \Phi(t,t_0)x_0 + \psi(t,t_0)q_0 \tag{10}$$

where

$$\psi(t,t_0) = \int_{t_0}^{t} \Phi(t,s)B(s)\tilde{B}'(s)\tilde{\Phi}'(T,s)ds$$
$$q_0 = \tilde{W}(T,t_0)^{-1}[\tilde{x}_d - \tilde{\Phi}(T,t_0)\tilde{x}_0] \tag{11}$$

A measure of the performance degradation in the original system caused by the suboptimal control is given by

$$J = \|\hat{x} - x^*\|^2 = \int_{t_0}^{T} [\hat{x}(s) - x^*(s)]'[\hat{x}(s) - x^*(s)]ds \tag{12}$$

A model will be termed optimal if it minimizes the worst possible performance deterioration when all permissible boundary conditions (x_0,x_d) are considered; i.e., a model which realizes $\tilde{J}$ is sought where

$$\tilde{J} = \min_{[\tilde{A},\tilde{B}]} \max_{(x_0,x_d)} \min_{(\tilde{x}_0,\tilde{x}_d)} \|\hat{x} - x^*\|^2 \tag{13}$$

Phrased in this way, the problem of modeling is seen to be a curve fitting problem in which the optimal trajectory $\hat{x}$ is approximated by a vector valued time function x^*, which is subject to the constraints given by Eqs. (1) and (8). The accuracy of the fit is measured by Eq. (13). From Eq. (13) it is seen that the parameters with respect to which the fit is accomplished are the elements of the $(m \times m + r + 2)$ matrix $[\tilde{A},\tilde{B},\tilde{x}_0,\tilde{x}_d]$.

3. Optimal Modelling

The evaluation of the optimal model may be decomposed into two parts. First, the initial and terminal conditions on the model must be found. Suppose that $[\tilde{A},\tilde{B}]$ and (x_0,x_d) are specified. Define

$$\hat{\zeta}(t) = \hat{x}(t) - \Phi(t,t_0)x_0,\ \zeta^*(t) = x^*(t) - \Phi(t,t_0)x_0 \tag{14}$$

Clearly,

$$\hat{x} - x^* = \hat{\zeta} - \zeta^* \tag{15}$$

A classical result of Hilbert space theory implies that J will be minimized when ζ^* is selected as the projection of $\hat{\zeta}$ on the subspace in which ζ^* must lie.[1,4] To determine this projection observe that

$$\hat{\zeta}(t) = W(t,t_0)p_0 \quad (16)$$

and $\zeta^*(t) = \psi(t,t_0)q_0$. Denote the columns of $\psi(t,t_0)$ by $\psi_i(t,t_0)$, $i = 1, \ldots, m$. Then, the projection ζ^* must satisfy

$$\int_{t_0}^{T} \psi_i'(s,t_0)\hat{\zeta}(s)ds = \int_{t_0}^{T} \psi_i'(s,t_0)\zeta^*(s)ds; \quad i = 1, \ldots, m \quad (17)$$

In matrix form Eqs. (16) and (17) become

$$\int_{t_0}^{T} \psi'(s,t_0)W(s,t_0)dsp_0 = \int_{t_0}^{T} \psi'(s,t_0)\psi(s,t_0)dsq_0 \quad (18)$$

Define

$$F = \int_{t_0}^{T} \psi'(s,t_0)W(s,t_0)ds, G = \int_{t_0}^{T} \psi'(s,t_0)\psi(s,t_0)ds \quad (19)$$

Then

$$Fp_0 = Gq_0 \quad (20)$$

Equation (20) gives an explicit relation between the given pair (x_0,x_d) and the optimal choice of the pair $(\tilde{x}_0,\tilde{x}_d)$. Further simplification of the problem can be made if the following condition is placed on the system of Eq. (1): if $\|x_0\| = 0$ and $\|u\| > 0$, then $\|x\| > 0$. This restriction is not strictly necessary for the development which follows but does significantly simplify the calculations.

From Eqs. (10) and (19) it follows that

$$\|\zeta^*\|^2 = q_0'Gq_0 \quad (21)$$

If $\|q_0\| \neq 0$, the invertibility of $\Phi(T,t)$ and the optimality of $\tilde{u}$ imply that $\|\zeta^*\| > 0$. Consequently, G is positive definite and G^{-1} exists. From Eq. (20) then[4]

$$q_0 = G^{-1}Fp_0 \quad (22)$$

Consequently, for a specified choice of (x_0,x_d), an optimal pair $(\tilde{x}_0,\tilde{x}_d)$ is

$$\tilde{x}_0 = 0, \tilde{x}_d = \tilde{W}(T,t_0)G^{-1}FW(T,t_0)^{-1}[x_d - \Phi(T,t_0)x_0] \quad (23)$$

Equation (23) provides a solution for part of the modeling problem. For a given set of model dynamics and specified (x_0,x_d), Eq. (23) gives the necessary terminal conditions on the model. Define

$$M = \int_{t_0}^{T} W'(s,t_0)W(s,t_0)ds \quad (24)$$

From Eqs. (16, 21, 22 and 24)

$$\min_{(\tilde{A},\tilde{B})} \max_{(x_0,x_d)} \min_{(\tilde{x}_0,\tilde{x}_d)} \|\tilde{x} - x^*\|^2 = \min_{(\tilde{A},\tilde{B})} \max_{(x_0,x_d)} p_0'(M - F'G^{-1}F)p_0 \quad (25)$$

The problem of maximizing $p_0'(M - F'G^{-1}F)p_0$ with respect to p_0 is a topic discussed in some detail in Ref. 4. The maximum value will clearly depend upon the set of allowable values for p_0. If this set is closed and convex, use may be made of the propositions stated in Ref. 4. In particular, suppose (x_0,x_d) is constrained by the condition

$$[x_d - \Phi(T,t_0)x_0]'H[x_d - \Phi(T,t_0)x_0] \leq r^2 \quad (26)$$

where H is non-negative definite and symmetric. From Eq. (5) and the theorem of Sec. 4 of Ref. 4 it is clear that the maximum of $p_0'(M - F'G^{-1}F)p_0$ will occur when

$$p_0'W'(T,t_0)HW(T,t_0)p_0 = r^2 \quad (27)$$

The resulting maximum is

$$\max_{(x_0,x)} \min_{(\tilde{x}_0,\tilde{x}_d)} \|\tilde{x} - x^*\|^2 = r^2\lambda_{\max}\{W(T,t_0)^{-1}H^{-1}W'(T,t_0)^{-1} \times [M - F'G^{-1}F]\} \quad (28)$$

where for any square matrix $A, \lambda_{\max}(A)$ is the eigenvalue of greatest magnitude. Consequently,

$$\tilde{J} = \min_{(\tilde{A},\tilde{B})} r^2\lambda_{\max}\{W(T,t_0)^{-1}H^{-1}W(T,t_0)^{-1} \times [M - F'G^{-1}F]\} \quad (29)$$

Equation (29) reduces the modeling problem to one of minimizing a non-negative scalar function of the $(m^2 + mr)$ elements of the $[\tilde{A},\tilde{B}]$ matrix. Unfortunately, because of the way in which the coefficient matrices enter into the performance measure it is not possible to display the dynamical equations of the optimal model explicitly. The final minimization indicated in Eq. (29) must be performed using a gradient search technique.

It is interesting to note that this minimum energy control problem has been reduced to a form which is computationally identical to that obtained by Heffes and Sarachik.[4] To perform the final minimization, Heffes found applicable a modified version of the Powell algorithm for minimizing a function of several variables without calculating derivatives.[5] This technique would appear to offer promise in this situation as well.

4. Conclusion

The problem of finding an optimal lower order model for a minimum energy control problem has been reduced to the minimization of a specific algebraic functional [see Eq. (25)]. If only one set of boundary conditions (x_0,x_d) is of interest, p_0 is fixed and the maximization with respect to (x_0,x_d) in Eq. (25) is trivial. It is much more complicated, however, to determine the best model if it is to be used for a range of initial and terminal conditions. When the class of allowable (x_0,x_d) are constrained by Eq. (26), some simplification can be achieved [see Eq. (29)]. Observe that the terminal conditions for the model given in Eq. (23) are valid without reference to the set of permissible (x_0,x_d). The controller resulting from this model is much simpler in form than the true minimum energy controller. As indicated in Eq. (8) it requires only linear algebraic operations on a matrix exponential.

The primary limitation on this proposed modeling technique is the computation complexity involved in performing the necessary minimization in Eq. (25). While this is certainly a restriction, it should be noted that these computations are done off-line and yield a model, which may then be utilized to provide controllers for many different applications.

The accuracy of the suboptimal system is of interest in selecting an appropriate value of m in Eq. (7). Bounds on performance were investigated in Ref. 4 and a very slight modification of the proof of Theorem 1 will show that if $p_0'p_0 \leq \rho^2$, then $\tilde{J} \geq \rho^2\lambda_{k+1}(M)$ where $\lambda_k(M)$ is the kth largest eigenvalue of M. The nearness with which the lower bound may be approached with any system of the form of Eq. (7) is not obvious a priori, but this lower bound does give a coarse measure of the rate at which performance is impaired by reducing the dimension of the state of the model.

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